JOURNAL OF APPROXIMATION THEORY 44, 221–229 (1985)

Best L_1-Approximation of Quasi Continuous Functions on [0, 1] by Nondecreasing Functions

ROBERT HUOTARI

Department of Mathematical Sciences, Indiana University–Purdue University, Fort Wayne, Indiana 46805 U.S.A.

Communicated by G. Meinardus

Received November 10, 1983

Let Q denote the Banach space (sup norm) of quasi continuous functions defined on the interval [0, 1]. Let M denote the closed convex cone in Q comprised of nondecreasing functions. For f in Q and $1 < p < \infty$, let f_p denote the best L_p-approximation to f by elements of M. It is shown that f_p converges uniformly as $p \to 1$ to a best L_1-approximation to f by elements of M. An example is given to show that this result is not true for all bounded measurable functions on [0, 1].

If f is a bounded Lebesgue measurable function defined on [0, 1] and A is a subset of $L_\infty[0, 1]$ such that, for each p, $1 < p < \infty$, there exists a unique best L_p-approximation f_p to f by elements of A, then f is said to have the Polya property if $f_\infty = \lim_{p \to \infty} f_p$ is well defined as a bounded measurable function: if $p_n \to \infty$, then $\lim_n f_{p_n}$ exists a.e. on [0, 1]. This limit is known to exist in a number of situations, and in each case the limit function is a best L_∞-approximation which is better in some way than all other best L_∞-approximations. Some of the investigations into the existence and the nature of this limit may be seen in [1–8]. A related question concerns the limit as $p \to 1$. f is said to have the Polya-one property if $f_1 = \lim_{p \downarrow 1} f_p$ is well defined as a bounded measurable function. In [6] it was shown that the Polya-one property obtains in the case where f is bounded and approximately continuous and A is the set of nondecreasing functions. In the present paper we establish the same result in the case where f is any quasi continuous function. We begin by showing that the Polya-one property holds if f is a real valued function with finite domain.

Let $X = \{x_1, ..., x_n\}$ be a finite subset of $\mathbb{R}$ with $x_1 < x_2 < \cdots < x_n$. Let $V = V(x)$ be the linear space of bounded real functions on X and $M_n = M(X) \subset V$ the convex cone of nondecreasing functions in V, i.e., functions h

0021-9045/85 $3.00

satisfying $h(x) \leqslant h(y)$ whenever $x, y \in X$ and $x \leqslant y$. For each p, $1 \leqslant p < \infty$, define a weighted l_p-norm $\|\cdot\|_w^p$ by

$$\|f\|_w^p = \left(\sum_{i=1}^{n} w_i |f_i| p \right)^{1/p}$$

where $f \in V$ is identified with its set of values $\{f(x_i): i = 1, ..., n\}$, denoted by $\{f_i\}$, and $w = \{w_i: i = 1, ..., n\} > 0$ is a given weight function satisfying $\sum_{i=1}^n w_i = 1$.

Let $f = \{f_1\}$ in V be fixed. For each p, $1 \leqslant p < \infty$, denote by P_p the following optimization problem: find $g_p = \{g_{p,i}: i = 1, ..., n\}$ in M_n, if one exists, such that

$$\|f - g_p\|_w^p = \inf\{\|f - h\|_w^p: h \in M_n\}.$$

To describe the known solutions to these problems, we first define $L \subset X$ to be a *lower set* if $x_i \in L$ and $x_j \in X$, $x_j \leqslant x_i$, implies that $x_j \in L$. Similarly, we call $U \subset X$ an *upper set* if $x_i \in U$ and $x_j \in X$, $x_j \geqslant x_i$ implies that $x_j \in U$. To simplify the notation we will write $i \in Y \subset X$ to indicate that $x_i \in Y$. Let p in $(1, \infty)$ be fixed. Let L and U be lower and upper sets, respectively, such that $L \cap U$ is not empty. Define u_p $(L \cap U)$ to be the unique real number minimizing $\sum \{w_j |f_j - u|^p: j \in L \cap U\}$. Let $g_p = \{g_{p,i}: i = 1, ..., n\}$ be the function defined on X by

$$g_{p,i} = \max_{\{U: i \in U\}} \min_{\{L: i \in L\}} u_p(L \cap U). \tag{1}$$

The solution of the problem P_p for $1 < p < \infty$ is known to be given by (1) (see [11]). Ubhaya [10] studied the convergence of g_p as $p \to \infty$. Our first objective in this paper is to show that convergence also results if p is allowed to decrease to one.

LEMMA 1. *Suppose $[a, b] \subset \mathbb{R}$ and $F = \{f_\lambda: \lambda \in \Lambda\}$ is a family of strictly convex functions on $\mathbb{R}$ such that, for all λ in Λ, the minimizer, x_λ, of f_λ is contained in (a, b). Define $\psi: (F, \|\cdot\|_\infty) \to \mathbb{R}$ by $\psi(f_\lambda) = x_\lambda$. Then ψ is continuous.*

Proof. Let f_1 in F and $\alpha < \max\{x_1 - a, b - x_1\}$ be given. Let $2\beta = \min\{f_1(x_1 - \alpha) - f_1(x_1), f_1(x_1 + \alpha) - f_1(x_1)\}$. Then $|x - x_1| \geqslant \alpha$ implies that $f_1(x) \geqslant f_1(x_1) + 2\beta$. Suppose that $\max\{|f_1(x) - f_2(x)|: x \in (a, b)\} < \beta$. If $|x_2 - x_1| \geqslant \alpha$, then

$$f_2(x_1) > f_2(x_2) > f_1(x_2) - \beta > f_1(x_1) + \beta,$$

a contradiction. Thus $|x_2 - x_1| < \alpha$.

DEFINITION. Let $a = -\|f\|_\infty$, $b = \|f\|_\infty$ and define functions τ_p: $[a, b]^n \to \mathbb{R}$ and κ_p: $[a, b] \to \mathbb{R}$ for $1 \leqslant p < \infty$ by

$$\tau_p(\mathbf{u}) = \sum_{i=1}^{n} w_i |f_i - u_i|^p,$$

$$\kappa_p(u) = \sum_{i=1}^{n} w_i |f_i - u|^p,$$

where $\mathbf{u} = (u_1, ..., u_n) \in [a, b]^n$ and $u \in [a, b]$.

LEMMA 2. *For each p, $1 < p < \infty$, κ_p is strictly convex and has a unique minimizer u_p, with u_p in $[a, b]$.*

Proof. Whenever $1 < p < \infty$ and $1 \leqslant i \leqslant n$, $|f_i - u|^p$ is a strictly convex function of u. Since $w > 0$, κ_p is also strictly convex, which entails the existence and uniqueness of u_p. It is clear that $a \leqslant u_p \leqslant b$.

LEMMA 3. *In the present context,*

$$\lim_{p \downarrow 1} (\tau_p(\mathbf{u}))^{1/p} = \tau_1(\mathbf{u})$$

and

$$\lim_{p \downarrow 1} (\kappa_p(u))^{1/p} = \kappa_1(u),$$

the convergence being uniform on the compact sets $[a, b]^n$ and $[a, b]$ respectively.

Proof. Whenever $u \in [a, b]^n$, $1 \leqslant i \leqslant n$ and $p < 2$,

$$|f_i - u_i|^p \leqslant 2^p\{|f_i|^p + |u_i|^p\} \leqslant 2^{p+1} \|f\|_\infty^p \leqslant m(f),$$

where $m(f) = 2^3 \max\{\|f\|_\infty^2, 1\}$. Let $\varepsilon > 0$ be given. For any $\mathbf{u}$ in $[a, b]^n$ and $0 < \alpha < 1$,

$$\begin{aligned} &|\tau_{1+\alpha}(\mathbf{u})^{1/(1+\alpha)} - \tau_1(\mathbf{u})| \\ &\quad\leqslant \left| \left\{ \sum_{i=1}^{n} w_i |f_i - u_i|^{1+\alpha} \right\}^{1/(1+\alpha)} - \left\{ \sum_{i=1}^{n} w_i |f_i - u_i| \right\}^{1/(1+\alpha)} \right| \qquad (2) \\ &\quad\quad + \left| \left\{ \sum_{i=1}^{n} w_i |f_i - u_i| \right\}^{1/(1+\alpha)} - \sum_{i=1}^{n} w_i |f_i - u_i| \right|. \end{aligned}$$

Since the map $x \mapsto x^{1/(1+\alpha)}$ is continuous for $x \geqslant 0$, there exists $\delta > 0$ such that the first summand of (2) is less than $\varepsilon/2$ whenever

$$|\tau_{1+\alpha}(\mathbf{u}) - \tau_1(\mathbf{u})| < \delta. \tag{3}$$

To see that there is an α small enough to satisfy (3), consider the function $F(x, \alpha) = x^{1+\alpha} - x$. Then $\partial F/\partial x = (1+\alpha)\, x^{\alpha} - 1$, $\partial F/\partial x = 0$ only when $x = x_0 = (1+\alpha)^{-1/\alpha}$ and $F(x_0) = (1+\alpha)^{-(1+1/\alpha)} - (1+\alpha)^{-1/\alpha}$. Let

$$B(\alpha) = 2 \max\{|F(x_0, \alpha)|, |[m(f)]^{1+\alpha} - m(f)|\}.$$

Then $\sup\{|F(x, \alpha)| : 0 < x < m(f)\} < B(\alpha)$, so for $\mathbf{u}$ in $[a, b]^n$ and $1 \leqslant i \leqslant n$,

$$|f_1 - u_i|^{1+\alpha} - |f_i - u_i| < B(\alpha).$$

Thus

$$|\tau_{1+\alpha}(\mathbf{u}) - \tau_1(\mathbf{u})| \leqslant \sum_{i=1}^{n} w_i \big|\, |f_i - u_i|^{1+\alpha} - |f_i - u_i| \,\big|$$

$$\leqslant B(\alpha) \sum_{i=1}^{n} w_i = B(\alpha).$$

Since $\lim_{\alpha \downarrow 0} F(x_0, \alpha) = 0$, it is clear that there exists $\alpha_0 > 0$ such that, for $0 < \alpha < \alpha_0$, $B(\alpha) < \delta$. This establishes (3).

To treat the second summand of (2), let $x = \sum_{i=1}^{n} w_i |f_i - u_i|$. Then $0 < x < \sum w_i 2\|f\|_\infty - 2\|f\|_\infty$. Define G by

$$G(x, \beta) = x^{1/(1+\beta)} - x.$$

Then $\partial G/\partial x = (1+\beta)^{-1} x^{-\beta/(1+\beta)} - 1$, $\partial G/\partial x = 0$ only when $x = x_0 = (1+\beta)^{-(1+1/\beta)}$ and $G(x_0, \beta) = (1+\beta)^{-1/\beta} - (1+\beta)^{-(1+1/\beta)}$. Since $G(x_0, \alpha) = -F(x_0, \alpha)$, the device of the previous paragraph shows that there exists $\beta_0 > 0$ such that, for $0 < \beta < \beta_0$,

$$|x^{1/(1+\beta)} - x| < \varepsilon/2.$$

Let $\gamma_0 = \min\{\alpha_0, \beta_0\}$. Then, for $0 < \gamma < \gamma_0$, and for any $\mathbf{u}$ in $[a, b]^n$,

$$|\tau_{1+\gamma}(\mathbf{u})^{1/(1+\gamma)} - \tau_1(\mathbf{u})| < \varepsilon. \tag{4}$$

The second limit follows from the first if we let $\mathbf{u} = (u, u, ..., u)$. This concludes the proof of Lemma 3.

A consequence of the proof of Lemma 3 may be noted at this time: For $1 \leqslant p < \infty$, let

$$d_n(p) = \inf\{\|f - \mathbf{u}\|_w^p : \mathbf{u} \varepsilon M_n\} = \inf\{\|f - \mathbf{u}\|_w^p : u \in M_n \cap [a, b]^n\}.$$

Then

$$\lim_{p\downarrow 1} d_n(p) = d_n(1). \tag{5}$$

Indeed, from (4), we see that, for all $\varepsilon > 0$, there exists $\gamma_0 > 0$ such that, for $0 < \gamma < \gamma_0$,

$$|\, \|f-\mathbf{u}\|_w^{1+\gamma} - \|f-\mathbf{u}\|_w^1 \,| < \varepsilon.$$

Then

$$\begin{aligned} &\inf\{\|f-\mathbf{u}\|_w^1 - \varepsilon : \mathbf{u} \in M_n \cap [a,b]^n\} \\ &\quad < \inf\{\|f-\mathbf{u}\|_w^{1+\gamma} : \mathbf{u} \in M_n \cap [a,b]^n\} \\ &\quad < \inf\{\|f-\mathbf{u}\|_w^1 + \varepsilon : \mathbf{u} \in M_n \cap [a,b]^n\}; \end{aligned}$$

so $|d_n(1+\gamma) - d_n(1)| < \varepsilon$. That a similar statement holds for $d(p) = \inf\{\|f-u\|_w^p : u \in R\}$ can be seen by letting $\mathbf{u} = (u, u, ..., u)$ in (5).

THEOREM 4. *For $1 < p < \infty$, let u_p be the unique minimizer of κ_p. Then $\lim_{p\downarrow 1} u_p$ exists. If $u_1 = \lim_{p\downarrow 1} u_p$, then u_1 is a minimizer of κ_1.*

Proof. By Lemma 2, $\{\kappa_p : 1 < p < \infty\}$ is a family of strictly convex functions on R with each u_p in $[a, b]$. Thus, by Lemma 1, $\alpha > 0$ and $1 < q < \infty$ implies that there exists $\beta(\kappa_q, \alpha) > 0$ such that, for $1 < r < \infty$ and $\max\{|\kappa_q(u) - \kappa_r(u)| : u \in [a,b]\} < \beta(\kappa_q, \alpha)$ we have $|u_q - u_r| < \alpha$. By reasoning similar to that establishing (3), $\kappa_p \to \kappa_q$ uniformly on $[a, b]$ as $p \to q$ so there exists $\delta > 0$ such that, for $|q - r| < \delta$,

$$\max\{|\kappa_q(u) - \kappa_r(u)| : u \in [a,b]\} < B(\kappa_q, \alpha).$$

Thus, the map $p \mapsto u_p$ is right continuous on $(1, \infty)$. Similarly, $p \mapsto u_p$ is left continuous. Suppose $\lim_{p\downarrow 1} u_p$ does not exist. Let $v' = \underline{\lim}_{p\downarrow 1} u_p$ and $v'' = \overline{\lim}_{p\downarrow 1} u_p$. Choose u_0 so that $v' < u_0 < v''$ and, for $1 \leqslant i \leqslant n$, $f_i - u_0 \neq 0$. Since $p \mapsto u_p$ is continuous, there exists an infinite sequence $\{p_k\}$ such that $p_k \downarrow 1$ and, for all $k \geqslant 1$, $u_{pk} = u_0$. Consider the function

$$F(p) = \kappa_p'(u_0) = p \sum_{i=1}^{n} w_i |f_i - u_0|^{p-1} \operatorname{sgn}(f_i - u_0).$$

For all $k \geqslant 1$, $F(p_k) = 0$ so 1 is a limit point of the set of zeros of F. Since $F(z)$ is entire, it is identically zero, whence $u_p = u_0$ for all $p > 1$, a contradiction. Thus $\lim_{p\downarrow 1} u_p$ exists.

Since $\sum w_i = 1$, we can apply inequality (2.10.4) in [9]: for any $p > 1$,

$$d(1) \leqslant \|f - u_p\|_w^1 \leqslant \|f - u_p\|_w^p.$$

Since $d(p) \to d(1)$, by (5), and $u_p \to u_1$, by the previous paragraph, $\|f-u_1\|_w^1 = d(1)$, whence u_1 is a minimizer of κ_1.

THEOREM 5. *The solution* $g_p = \{g_{p,i}: i=1,..., n\}$ *of the problem* P_p *converges as* $p \downarrow 1$ *to a solution*

$$g_1 = \{g_{1,i}: i=1,..., n\} \tag{6}$$

of the problem P_1.

Proof. The solution, g_p, of the problem P_p, $1 < p < \infty$, is given by (1). Considering $L \cap U$ instead of X in Theorem 4, we conclude that $\lim_{p \downarrow 1} u_p(L \cap U)$ exists. Let $u_1(L \cap U)$ denote this limit. Since the number of lower and upper sets is finite, from (1) it follows that the limit of $g_{p,i}$ exists as $p \downarrow 1$ for all i. It remains to be shown that g_1 is a solution of the problem P_1. Since g_p is nondecreasing for each $p > 1$, g_1 also has this property.

As in the proof of Theorem 4, we have

$$d_n(1) \leqslant \|f-g_p\|_w^1 \leqslant \|f-g_p\|_w^p.$$

Since $d_n(p) \to d_n(1)$ by (5), and $g_p \to g_1$ by the previous paragraph,

$$\|f-g_1\|_w^1 = d_n(1),$$

whence g_1 is a solution to the problem P_1. This concludes the proof of Theorem 5, and accomplishes our first objective.

A function $f: [0, 1] \to \mathbb{R}$ is said to be *quasi continuous* if it has discontinuities of the first kind only. Let Q denote the set of all quasi continuous functions. Our goal in the remainder of this paper is to generalize Theorem 5 to the case where $f \in Q$.

Let P denote the set of partitions $\pi = \{t_i: i=0, 1,..., n\}$ of $[0, 1]$ (i.e., $0 = t_0 < t_1 < \cdots < t_n = 1$), let I_E denote the indicator function of a subset E of $[0, 1]$ (i.e., $I_E(x) = 1$ if x is in E and $I_E(x) = 0$ otherwise), and let S denote the dense linear subspace of Q comprised of simple step functions of the form

$$f = \sum_{i=0}^{n} a_i I_{[t_i]} + \sum_{i=1}^{n} b_i I_{(t_{i-1}, t_i)}.$$

For a subset A of Q, let A^* denote the set of left continuous elements of A. Then f is in S^* if there exists π in P such that

$$f = a_1 I_{[t_0, t_1]} + \sum_{i>1} a_i I_{(t_{i-1}, t_i]}.$$

For a bounded function f and π in P, $\bar{f}_\pi$ in S^* is defined by

$$\begin{aligned} \bar{f}_\pi(x) &= \sup\{f(y): y \in [t_0, t_1]\}, \qquad x \in [t_0, t_1] \\ &= \sup\{f(y): y \in (t_{i-1}, t_i]\}, \qquad x \in (t_{i-1}, t_i], i > 1, \end{aligned}$$

$\underline{f}_\pi$ is defined by replacing sup with inf.

A bounded function f is in Q^* if and only if, for any $\varepsilon > 0$, there exists π in P such that $0 \leqslant \bar{f}_\pi - \underline{f}_\pi < \varepsilon$. This allows the use of Theorem 5.

Because L_p is a uniformly convex Banach space, $1 < p < \infty$, for each f in Q^* there exists a unique nearest point f_p in M^*. We recall the following result of [8].

THEOREM 6. *Let f in S_π^* be given by*

$$f = f_1 I_{[0,t_1]} + \sum_{i=2}^{n} f_i I_{(t_{i-1}, t_i]}.$$

Let $w = \{w_i: i = 1,..., n\}$ be defined by $w_i = t_i - t_{i-1}$ for all i. For $1 < p < \infty$, let g_p be as defined by (1). *Then f_p is given by*

$$f_p = g_{p,1} I_{[0,t_1]} + \sum_{i=2}^{n} g_{p,i} I_{(t_{i-1}, t_i]}.$$

The next theorem is a slightly altered form of Theorem 3 in [8].

THEOREM 7. *Let f in S_π^* and f_p be as given in Theorem* 6. *Then f_p converges as $p \downarrow 1$ to the monotone increasing function f_1 in S_π^* given by*

$$f_1 = g_{1,1} I_{[0,t_1]} + \sum_{i=2}^{n} g_{1,i} I_{(t_{i-1}, t_i]}, \tag{7}$$

where $g_{1,i} = \lim_{p \downarrow 1} g_{p,i}$ is given by (6). *Moreover, f_1 is a best L_1-approximation to f by nondecreasing functions.*

Proof. For each i, $1 \leqslant i \leqslant n$, let $x_i = (t_i + t_{i-1})/2$ and let $X = \{x_1,..., x_n\}$. Consider $\{f_i = f(x_i): i = 1,..., n\}$ as a finite real valued function on X. Let w be defined as above. Then Theorem 5 implies that g_p converges to g_1. Therefore $\lim_{p \downarrow 1} f_p$ exists and is given by (7).

For the second part of the theorem, we note that the conclusion of Theorem 5 holds for any weight function $w = \{w_i: i = 1,..., n\}$ which satisfies the conditions $w > 0$ and $\sum w_i = 1$. For each i, $1 \leqslant i \leqslant n$, let $w_i = 1/n$; then Theorem 5 implies that

$$\sum_{i=1}^{n} n^{-1} |f_i - g_{1,i}| \leqslant \sum_{i=1}^{n} n^{-1} |f_i - h|, \qquad h = \{h_i: i = 1,..., n\} \in M_n,$$

whence

$$\sum_{i=1}^{n} |f_i - g_{1,i}| \leqslant \sum_{i=1}^{n} |f_i - h|, h \in M_n. \tag{8}$$

Thus, f_1 is a best L_1-approximation to f by elements of S_π^*. Let h be a nondecreasing function defined on $[0, 1]$. We show that there is a nondecreasing function g in S_π^* such that

$$\|f-g\|_1 \leqslant \|f-h\|_1.$$

Indeed, for each i, $1 \leqslant i \leqslant n$, let g_i be the real number in the interval $[h(t_{i-1}), h(t_i)]$ nearest to f_i. Then, for each i,

$$\int_{t_{i-1}}^{t_i} |f_i - g_i| \leqslant \int_{t_{i-1}}^{t_i} |f_i - h(x)|.$$

Now define g on $[0, 1]$ by

$$g = g_1 I_{[0,t_1]} + \sum_{i=2}^{n} g_i I_{(t_{i-1}, t_i]}.$$

Then g is in S_π^* and it follows from the last inequality together with (8) that

$$\|f-f_1\|_1 \leqslant \|f-g\|_1 \leqslant \|f-h\|_1.$$

This concludes the proof of Theorem 7.

The remainder of the proof in [8] is now easily adapted to yield our principal result.

THEOREM 8. *Let $f \in Q$. Then there exist nondecreasing functions f_p, $1 \leqslant p < \infty$, such that each f_p is (up to equivalence) a best L_p-approximation to f by nondecreasing functions and f_p converges uniformly to f_1 as p decreases to one.*

EXAMPLE 9. If g is bounded measurable function on an interval $[a, b]$, we say that g has the uniform Polya-one property if g_p converges uniformly as $p \to 1$ to a best L_1-approximation to g by elements of M. An example of a bounded measurable function on a compact interval which does not have the uniform Polya-one property is constructed as follows: for $n > 1$, let

$$a_n = \sum_{i=1}^{n-1} (2^{1-i} + 4^{-i}),$$

$$b_n = 2^{-n} + \sum_{i=1}^{n-1} (2^{1-i} + 4^{-i}),$$

$$A = [0, 1/2] \cup \bigcup_{n=2}^{\infty} [a_n, b_n],$$

and $g=I_A \mid [0, \frac{7}{3}]$. Since $m[g=0]>\frac{7}{6}$, $g_1 \equiv 0$. If $t>0$ and $n>1$ are given, let

$$F(x)=2^{-n}(1-x)^{1+t}+(2^{-n}+4^{-n})\,x^{1+t}.$$

Then $F'(x)=0$ implies that $x=x_0(t,n)=\{(1+2^{-n})^{1/t}+1\}^{-1}$, which is the value of g_{1+t} on the interval $[a_n, a_{n+1}]$. Since $x_0(t,n)$ increases to $\frac{1}{2}$ as $n\to\infty$, there exists N such that, for $n\geqslant N$, $x_0(t,n)>\frac{1}{4}$. Thus $\|g_{1+t}-g_1\|_\infty>\frac{1}{4}$ so g_{1+t} does not converge in L_∞ to g_1 as $t\downarrow 0$. Let

$$\begin{aligned} B=[0,\tfrac{7}{3}]&-(2^{-1}-4^{-3}, 2^{-1}+4^{-3}) \\ &-\bigcup_{n=2}^{\infty}\{(a_n-4^{-3n}, a_n+4^{-3n})\cup(b_n-4^{-3n}, b_n+4^{-3n})\}. \end{aligned}$$

Then $g \mid B$ may be extended to a function which is continuous on $[0, \frac{7}{3})$ and does not have the uniform Polya-one property.

References

1. A. M. Al-Rashed, Best L_∞-approximation in probability spaces, *J. Math. Anal. Appl.* **91** (1983), 1–8.
2. A. M. Al-Rashed and R. B. Darst, Convergence of best best L_∞-approximations, *Proc. Amer. Math. Soc.* **83** (1981), 690–692.
3. A. M. Al-Rashed and R. B. Darst, Best L_∞-approximation of measurable vector-valued functions, submitted for publication.
4. R. B. Darst, Convergence of L_p-approximations as $p\to\infty$, *Proc. Amer. Math. Soc.* **81** (1981), 433–436.
5. R. B. Darst, Approximation in L_p, $1<p<\infty$, *in* "Measure Theory and Applications, Proceedings, 1980 Conference, Northern Illinois University" (G. A. Golden and R. F. Wheeler, Eds.), De Kalb, Ill., 1981.
6. R. B. Darst and R. Huotari, Best L_1-approximation of bounded. approximately continuous functions in $[0, 1]$ by nondecreasing functions, *J. Approx. Theory* **43** (1985), 178–189.
7. R. B. Darst, D. A. Legg, and D. W. Townsend, The Polya algorithm in L_∞-approximation, *J. Approx. Theory* **38** (1983), 209–220.
8. R. B. Darst and S. Sahab, Approximation of continuous and quasicontinuous functions by monotone functions, *J. Approx. Theory*, **38** (1983), 9–27.
9. G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities," Cambridge, 1934.
10. V. A. Ubhaya, Isotone optimization II, *J. Approx. Theory*, **12** (1974), 146–159.
11. C. Van Eeden, "Testing and Estimating Ordered Parameters of Probability Distributions," Ph.D. dissertation, University of Amsterdam, Studentendrukkerij Poortpers N.V., Amsterdam, 1958.